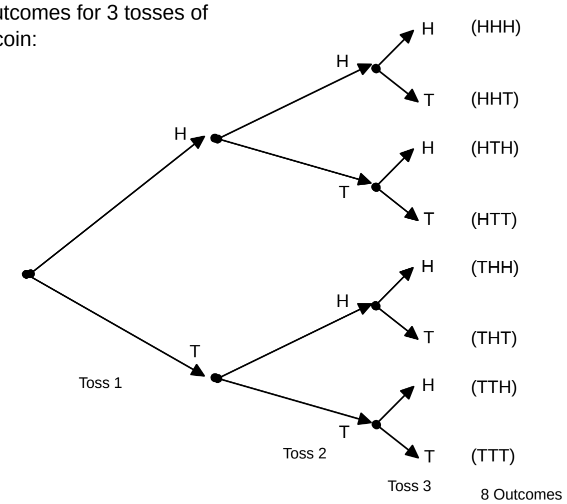
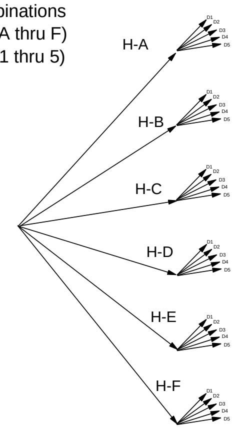
Outcomes for 3 tosses of a coin:

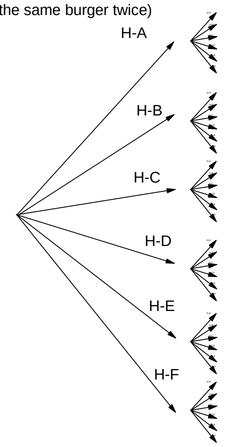


How many combinations of hamburgers (A thru F) and soft drinks (1 thru 5) at Wendy's?



There are 6 x 5 = 30 possible combinations of hamburgers and soft drinks

How many different combinations of Wendy's hamburgers can we buy, if we buy 2 burgers? (One combination could be the same burger twice)



36 different combinations? (6x6=36?)

No! From the top branch we can get 6 combinations, but from the next branch only 5, the next only 4, 3, 2 and 1 = 21 combinations, allowing the purchase of 2 burgers of the same kind.

What if we don't allow 2 burgers of the same kind?

Then, we can only find

$$\binom{6}{2} = \frac{6!}{2!4!} = \frac{720}{2 \times 24} = 15 \text{ combinations}$$

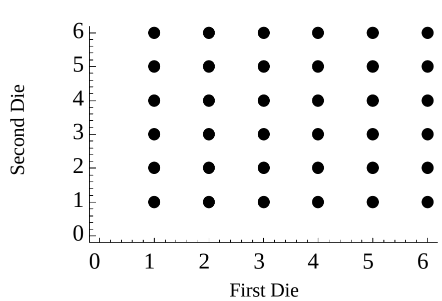
Number of Permutations of *n* things taken *r* at a time

$$_nP_r = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1)$$

Number of Combinations of *n* things taken *r* at a time

$$_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{_nP_r}{r!}$$

Sample Space for One Roll of Two Die



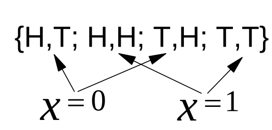
= 36 outcomes

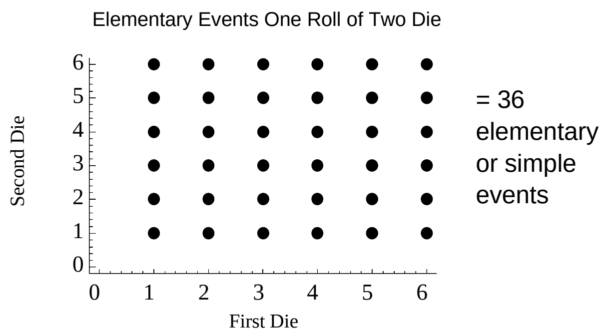
$\underset{\sim}{x} =$ 1 if outcome = H
0 if outcome = T

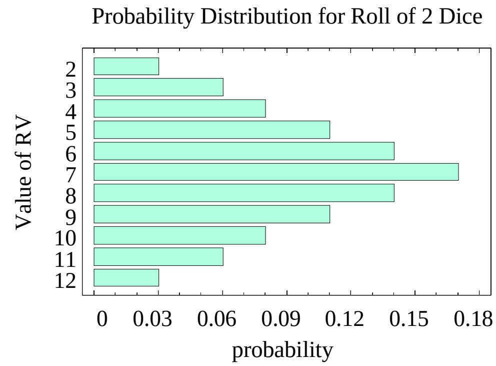
Toss a coin

$\underset{\sim}{x} =$ 1 if there is a match
0 if there is no match

Matching Pennies

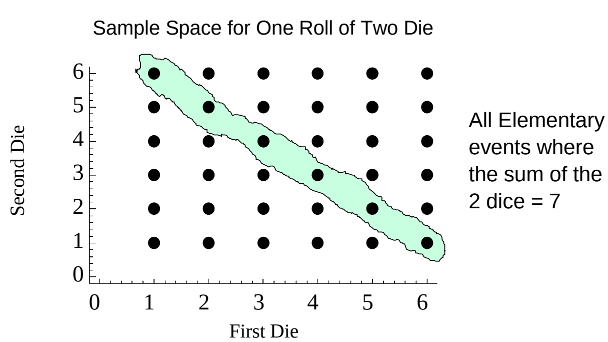


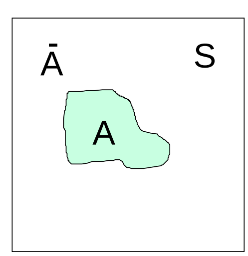




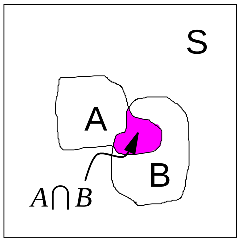
$$
\begin{array}{lllll}
S = & \{e_1, & e_2, \ldots & \ldots, & e_k\} \\
 & \downarrow & \downarrow & & \downarrow \\
 & p_1, & p_2, \ldots & \ldots, & p_k\}
\end{array}
$$

$$e_1 \rightarrow pr(e_1) = p_1$$
$$e_2 \rightarrow pr(e_2) = p_2$$
$$e_3 \rightarrow pr(e_3) = p_3$$
$$\ldots$$
$$e_k \rightarrow pr(e_k) = p_k$$



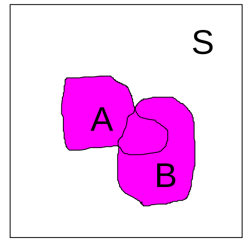


- The sample space, S, contains all the elementary events
- The Event A contains all elementary events corresponding to an inclusion rule.
- All other elementary events are in the complement A-bar.



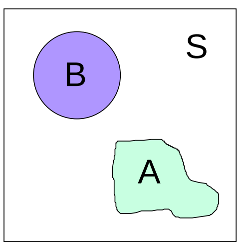
- The intersection consists of all outcomes common to two events.

$$A \cup B = \{\underset{\sim}{x} \mid x \in A \text{ or } x \in B\}$$



- The Union of two events A & B consists of all elementary events contained in either A or B.

$$\Pr(A \cup B) = P(A) + P(B) = \Pr(A \text{ or } B)$$

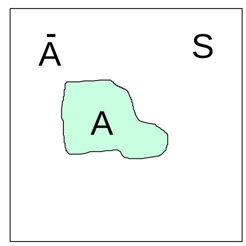


A and B are two mutually exclusive events (no elementary events in common),

Or, more generally:

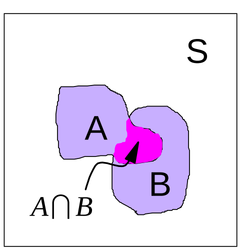
$$\Pr(A \cup B \cup C \cup \ldots) = P(A) + P(B) + P(C) + \ldots = \Pr(A \text{ or } B \text{ or } C \text{ or } \ldots)$$

$$\begin{aligned}\Pr\{A \cup \bar{A}\} &= \Pr(A) + \Pr(\bar{A}) \\ &= \Pr(A) + 1 - \Pr(A) \\ &= 1.\end{aligned}$$



- The Probability of the union of complementary events, A & A-bar is one!

$$\Pr(A \cup B) = P(A) + P(B) - P(A \cap B) = \Pr(A \text{ or } B)$$



The **General Addition Rule** requires the subtraction of the probability of Intersecting events.

or, more generally,

$$\Pr(A \cup B \cup C \cup \ldots) = P(A) + P(B) + P(C) + \ldots - P(A \cap B \cap C \cap \ldots) = \Pr(A \text{ or } B \text{ or } C \text{ or } \ldots)$$

Results of Antihistamine Experiment

	Received the Drug D	Received the Placebo $\bar{D}$	Total
Allergic Symptoms A	16	8	24
No Allergic Symptoms $\bar{A}$	64	12	76
Total	80	20	100

Contingency Table

Results of Antihistamine Experiment

	Received the Drug D	Received the Placebo $\overline{D}$	Total
Allergic Symptoms A	16	8	24
No Allergic Symptoms $\overline{A}$	64	12	76
Total	80	20	100

Cells

Totals

Joint Probability Distribution

	Received the Drug D	Received the Placebo $\overline{D}$	$P(A)$ & $P(\overline{A})$
Allergic Symptoms A	$P(A \cap D) = 0.16$	$P(A \cap \overline{D}) = 0.08$	$P(A) = 0.24$
No Allergic Symptoms $\overline{A}$	$P(\overline{A} \cap D) = 0.64$	$P(\overline{A} \cap \overline{D}) = 0.12$	$P(\overline{A}) = 0.76$
$P(D)$ & $P(\overline{D})$	$P(D) = 0.80$	$P(\overline{D}) = 0.20$	1.0

Joint Probabilities

Marginal Probabilities

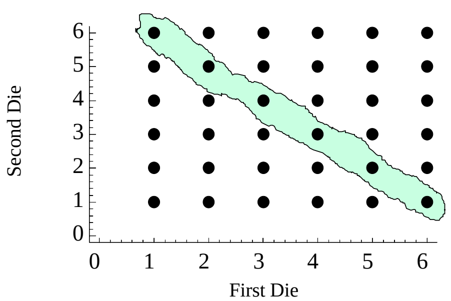
	Received the Drug D	Received the Placebo $\overline{D}$	$P(A)$ & $P(\overline{A})$
Allergic Symptoms A	$P(A \cap D) = 0.16$	$P(A \cap \overline{D}) = 0.08$	$P(A) = 0.24$
No Allergic Symptoms $\overline{A}$	$P(\overline{A} \cap D) = 0.64$	$P(\overline{A} \cap \overline{D}) = 0.12$	$P(\overline{A}) = 0.76$
$P(D)$ & $P(\overline{D})$	$P(D) = 0.80$	$P(\overline{D}) = 0.20$	1.0

The Conditional Probability Rule:
If A and B are any two events, then

$$P(B|A) = \frac{P(A \& B)}{P(A)}.$$

In words, for any two events the conditional probability that one event occurs given that the other even has occurred equals the joint probability of the two events divided by the probability of the given event.

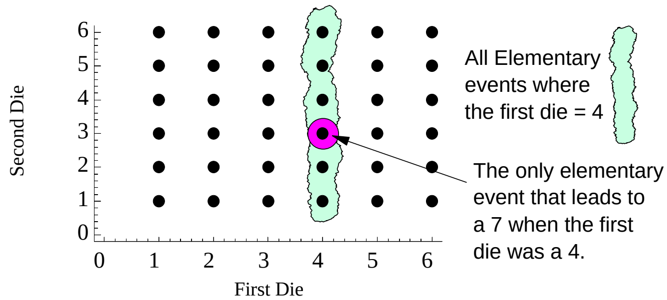
Sample Space for One Roll of Two Die



All Elementary events where the sum of the 2 dice = 7. (Event C)

P(C)= 6/36 = 1/6

Sample Space for One Roll of Two Die



P(C|Die 1=4)=1/6

Ways to Roll a 7

Ways to roll a 3 when Die 1=4

First Die (A)

Second Die (B)

	1	2	3	4	5	6	
1	0.028	0.028	0.028	0.028	0.028	0.028	0.167
2	0.028	0.028	0.028	0.028	0.028	0.028	0.167
3	0.028	0.028	0.028	0.028	0.028	0.028	0.167
4	0.028	0.028	0.028	0.028	0.028	0.028	0.167
5	0.028	0.028	0.028	0.028	0.028	0.028	0.167
6	0.028	0.028	0.028	0.028	0.028	0.028	0.167
	0.167	0.167	0.167	0.167	0.167	0.167	1.000

$$P(B|A) = \frac{P(A\& B)}{P(A)} = \frac{P(B=3 \cap A=4)}{P(A=4)} = \frac{0.028}{0.167} = 0.167 = P(B=3) = P(C=7).$$

The General Multiplication Rule:

If A and B are any two events then

$$P(B\& A) = P(B) \bullet P(A|B).$$

In words, for any two events, their joint probability equals the probability that one of the events occurs times the conditional probability of the other event, given that event.

Independent Events

Event A is said to be **independent** of event B if the occurrence of event B does not affect the probability that event A occurs. In symbols,

$$P(A|B) = P(A).$$

In words, knowing whether event B has occurred provides no probabilistic information about the occurrence of event A.

Independence Implies

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = P(A) = P(A|S)$$

$$P(B|A) = \frac{P(B\cap A)}{P(A)} = P(B) = P(B|S)$$

Rearranging these two equations gives the General Multiplication Rule:

$$P(A\cap B) = P(A|B)P(B) \text{ or}$$

$$P(B\cap A) = P(B|A)P(A)$$

which implies, since

$$P(A\cap B) = P(B\cap A) \Rightarrow P(A|B)P(B) = P(B|A)P(A)$$

that, if A and B are independent events, then from the definition of independence:

$$P(A\cap B) = P(A|B)P(B) = P(A)\bullet P(B) \text{ or,}$$

$$P(B\cap A) = P(B|A)P(A) = P(B)\bullet P(A)$$

The Special Multiplication Rule (for Two Independent Events)

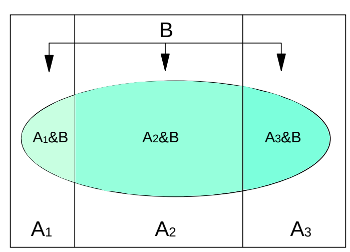
If *A* and *B* are independent events, then

$$P(B \,\&\, A) = P(B)\bullet P(A),$$

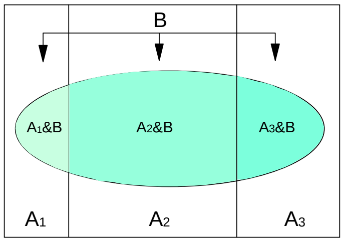
and conversely, if

$$P(B \,\&\, A) = P(B)\bullet P(A),$$

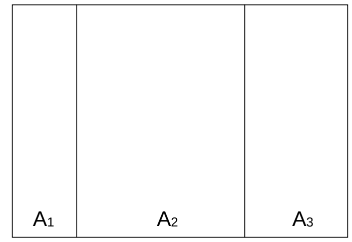
then *A* and *B* are independent events. In words, two events are independent if and only if their joint probability equals the product of their marginal probabilities.



Sample space, S, is completely filled by event A.



Sample space, S, is completely filled by event A.



Sample space, S, is completely filled by event A.

The Rule of Total Probability

Suppose that Events $A_1, A_2, \ldots, A_k$ are mutually exclusive and exhaustive; that is, exactly one of the events must occur. Then, for any event *B*,

$$P(B) = \sum_{j=1}^{k} P(A_j \cap B) = \sum_{j=1}^{k} P(A_j) \bullet P(B \mid A_j).$$

Bayes's Rule: Assume these probabilities are known(let $k = 3$):

$$P(A_1) \quad P(A_2) \quad P(A_3)$$
$$P(B|A_1) \quad P(B|A_2) \quad P(B|A_3)$$

Use these probabilities to find the following probabilities:

$$P(A_1|B) \quad P(A_2|B) \quad P(A_3|B)$$

From the Conditional Probability Rule, we know that:

$$P(A_i \mid B) = \frac{P(B \,\&\, A_i)}{P(B)} \text{ for } i = 1,3.$$

Next, apply the General Multiplication Rule to the Numerator:

$$P(A_i \mid B) = \frac{P(B \,\&\, A_i)}{P(B)} \; i = 1,3.$$

That gives us:

Prior probability

$$P(A_i \mid B) = \frac{P(A_i) \bullet P(B \mid A_i)}{P(B)} \; i = 1,3.$$

Posterior probability

Finally, apply the rule of Total Probability in the denominator to give us **Bayes's Rule:**

$$P(A_i \mid B) = \frac{P(A_i) \bullet P(B \mid A_i)}{\sum_{j=1}^{k} P(A_j) \bullet P(B \mid A_j)} \; i = 1,3$$